Star Formation in the Interstellar Medium
Workshop in Honor of David Hollenbach, Chris McKee, and Frank Shu
ASP Conference Series, Vol. ,
eds. F. C. Adams, D. Johnstone, D. N. C. Lin, & E. C. Ostriker

Symmetries, magnetic fields, and the shape of molecular cloud cores

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Abstract. We develop a formalism wherein the solution of the equations of equilibrium for a self-gravitating, magnetized interstellar cloud may be accomplished analytically, under the only hypotheses that (a) the cloud is scale-free, and (b) is electrically neutral. All variables are represented as series of scalar and vector spherical harmonics, and the equilibrium equations are reduced to a set of coupled algebraic equations for the expansion coefficients of the density and the magnetic vector potential. Previously known axisymmetric solutions are recovered and new non-axisymmetric solutions are found and discussed as models for pre-protostellar molecular cloud cores.

1. Antecedents

"Do you think that non-symmetric, magnetized, self-gravitating equilibria can exist?" This question is a typical example of the classic, fundamental kind of problems that Frank likes to address. This contribution presents a summary of the results obtained by the writer in his effort to answer Frank's question.

2. Introduction

Ignoring magnetic fields and considering only bounded masses, the result that static stars must be spherical probably follows from the expression of the Newtonian potential alone, but nobody has been able to give a rigorous proof (even less in general relativity, see Misner, Thorne, & Wheeler 1970). For a homogeneous static star, the spherical configuration is that of minimum gravitational energy (the proof is due to Lyapunov, and is reported in Poincaré 1902). Thus, the sphere is the only *stable* form of equilibrium. Lamb (1932) argued that "the only form of equilibrium of a mass of homogeneous liquid under its own attraction is a sphere" on the basis of the Lichtenstein (1918) theorem. As this theorem asserts, a rotating star is necessarily symmetric with respect to a plane perpendicular to the rotation axis and passing through the center of mass of the star; if there is no rotation, every plane through the center of mass must be a plane of symmetry, hence the star must be spherical.

Interstellar clouds are not homogeneous balls. The most idealized model of a cloud consists of an equilibrium configuration of an isothermal gas, in which self-gravity is balanced by thermal pressure alone. As it is well known, the equilibrium of such cloud is governed by the quasi-linear elliptic partial differential

equation (PDE)

$$\nabla^2\psi + e^\psi = 0, \tag{1}$$

where ψ is the non-dimensional gravitational potential. Does this equation allow only spherically symmetric solutions? With the condition $\psi = 0$ over an arbitrary boundary surface, eq. (1) is known to mathematicians as a Liouville-Gel'fand equation. In 1979, the fundamental theorem of Gidas, Ni, & Niremberg established that all solutions of this class of equations are spherically symmetric if the boundary surface is spherical.

The problem with real clouds is that, unlike stars, they have no "boundary surface": they have infinite extent on the scale of the forming star. When the boundary condition on eq. (1) is pushed to infinity, non-spherically symmetric solutions of eq. (1) appear, but they are either unphysical because they correspond to $\rho < 0$ (Matsuno 1987), or not force-free at singular points (Medvedev & Narajan 2000). Thus, the deceptively simple question of whether a balance of gravity and pressure alone may lead to physically acceptable non-spherical equilibria remains unsettled. In this contribution, we concentrate on dense, quiescent, molecular cloud cores, containing a few solar masses of gas, and characterized by very low levels of turbulence. A wealth of observations shows that cores are magnetized (see e.g. Crutcher 2001). Therefore, we formulate the problem of the existence of equilibrium states in the general context of magnetohydrostatics (MHS).

Little is known about the intrinsic shapes of cores, which are observed in projection against the plane of the sky. From a selected sample, Myers et al. (1991) concluded that the mean observed aspect ratio was consistent with either oblate or prolate spheroids. Later, for a larger sample, Ryden (1966) and Curry (2002) found that the observed distribution of axis ratios was more consistent with prolate rather than oblate shapes. Recent statistical studies (Jones, Basu, & Dubinski 2001; Jones & Basu 2002), based on the largest available catalogues of dense cores, found unsatisfactory agreement with either oblate or prolate spheroids, and rejected the possibility that cores are axisymmetric configurations. A good fit to the observed axial distribution was generally found assuming that cores are triaxial ellipsoids. The best-fit axial ratios $a : b : c$ and their standard deviations determined for the same sample by these studies are remarkably close: $1 : (0.9 \pm 0.1) : (0.5 \pm 0.1)$ according to Jones & Basu (2002), $1 : (0.8 \pm 0.2) : (0.4 \pm 0.2)$ according to Goodwin, Ward-Thompson, & Whitworth (2003). The result $a \approx b > c$ clearly suggests that cores are preferentially flattened in one direction and nearly oblate, and implies that molecular cloud cores may not be particularly far from conditions of equilibrium.

3. Ideal magnetohydrostatic equilibria

Consider a magnetized, isothermal, self-gravitating cloud satisfying the ideal MHS equations

$$c_{\rm s}^2\nabla\rho + \rho\nabla\mathcal{V} = \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{2}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{3}$$

$$\nabla^2\mathcal{V} = 4\pi G\rho, \tag{4}$$

where $\mathbf{B}$ is the magnetic field, ρ is the density, $\mathcal{V}$ is the gravitational potential, and c_s is the sound speed. All known solutions of this set of equations are characterized by a coordinate symmetry that reduces the number of variables from three to two or one. The symmetry associated with an ignorable coordinate allows MHS problems to be reduced to a second-order elliptic PDE (Dungey 1953), conventionally called the Grad-Shafranov equation. Symmetric equilibria may be grouped into (*i*) *axisymmetric* ($\partial/\partial\varphi = 0$), (*ii*) *cylindrically symmetric* ($\partial/\partial z = 0$), and (*iii*) *helically symmetric* systems ($\partial/\partial\varphi = k\,\partial/\partial z$). As shown by Edenstrasser (1980), helical symmetry represents the most general admissible invariance property of the MHS equations, with rotational and translational invariance as limiting cases.

Applications to the study of interstellar molecular clouds have largeley focused on magnetic configurations of the first kind, assuming azimuthal symmetry (e.g. Mouschovias 1976; Nakano 1979; Tomisaka, Ikeuchi,& Nakamura 1988). Cylindrically symmetric equilibria have been extensively studied, originally in connection with the stability of galactic spiral arms (Chandrasekhar & Fermi 1953; Stodółkiewicz 1963), and later as models of filamentary clouds (Nagasawa 1987; Nakamura, Hanawa, & Nakano 1995). In the latter context, magnetic configurations possessing helical symmetry have also been explored (Nakamura, Hanawa, & Nakano 1993, Fiege & Pudritz 2000a,b).

Even in the special case of non-selfgravitating plasmas, the fundamental question as to the existence of MHS equilibria of a more general symmetry or with no symmetry at all (3-D MHS equilibria) has been formulated several times (e.g. Low 1980, Degtyarev et al. 1985) but never properly answered. Grad (1967) conjectured that, with $\mathcal{V} = 0$, only "highly symmetric" solutions of the system (2)–(3) should be expected, in order to balance the highly anisotropic Lorentz force with with pressure gradients and gravity, which are forces involving scalar potentials. In a fundamental paper, Parker (1972) proved rigorously the non existence of 3-D MHS equilibria that are small perturbations of 2-D equilibria having translational symmetry. This result, conventionally referred to as *Parker's theorem*, stimulated much subsequent work and led to the concept of *topological non-equilibrium* for magnetic configurations lacking high degrees of symmetry (see e.g. Tsinganos, Distler, & Rosner 1984; Moffatt 1985, 1986; Vainshtein et al. 2000). Before extending Parker's theorem and its implications to conditions typical of molecular cloud cores, one must understand how (*a*) fluid motions and (*b*) gravitational forces affect the existence of equilibria.

As for the effect of fluid motions, Tsinganos (1982) found that Parker's theorem remains valid for steady dynamical ($\mathbf{v} \neq 0$) configurations possessing translational invariance, and stressed the analogy between this result and the familiar Taylor-Proudman theorem of hydrodynamics. However, somewhat at variance with these findings, Galli et al. (2001) found that a class of MHD equilibria possessing axial symmetry (rotating, magnetized, self-gravitating singular isothermal disks) does have neighboring non-axisymmetric equilibrium states, provided the rotation speed becomes sufficiently high (supermagnetosonic). As for the effect of gravity, Field (1982, quoted by Tsinganos et al. 1984) noted that the neglect of the constraining effect exerted by the plasma's self-gravity may severely limit the existence of MHS equilibria. In a series of papers (Low 1985; Bogdan & Low 1986; Low 1991), Low and collaborators have elaborated

a general method to solve the equations of MHS in the presence of an external gravitational field, either uniform or variying as $1/r^2$, without postulating the existence of an ignorable coordinate but assuming an electric current everywhere perpendicular to the gravitational field. Even under these restricted assumptions, the problem presents however considerable mathematical difficulties.

4. Scale-free equilibria

In order to simplify the problem, we assume that the density and magnetic field are scale-free, i.e. varying as powers of the radial coordinate r in spherical coordinates (r, θ, φ). In general, any solenoidal vector field $\mathbf{B}$ can be expressed as a linear combination of two basic fields $\mathbf{T}$ and $\mathbf{S}$ (Chandrasekhar 1961), characterized by vanishing radial component ($T_r = 0$) and vanishing radial component of the curl ($[\nabla \times \mathbf{S}]_r = 0$), respectively. For a scale-free configuration, if there is no current at one radius, then the same is true for all radii. This condition must be certainly satisfied at large radii, otherwise there would be a flow of charge to infinity, and the cloud would become electrically charged. Thus, the magnetic field in our problem must be purely of the $\mathbf{S}$-type, and given by

$$\mathbf{B} = \nabla \times \left[\nabla \left(\frac{\Psi}{r}\right) \times \mathbf{r}\right], \tag{5}$$

where $\Psi(r, \theta, \varphi)$ is an arbitrary scalar function of position. Here we adopt the scaling of Li & Shu (1996, hereafter LS96),

$$\rho = \frac{c_{\rm s}^2}{2\pi G r^2} R(\theta, \varphi), \tag{6}$$

$$\mathcal{V} = 2c_{\rm s}^2[(1 + H_0)\ln r + V(\theta, \varphi)], \tag{7}$$

$$\Psi = \frac{2c_{\rm s}^2 r}{\sqrt{G}} F(\theta, \varphi). \tag{8}$$

The quantity H_0 in the expression of the gravitational potential eq. (7) is a dimensionless constant measuring the fractional increase in the mean density that arises because the magnetic field contributes to support the cloud against self-gravity (LS96).

5. Spectral magnetohydrostatics

A natural basis for a multipole expansion of eq. (2)–(4) is provided by *scalar spherical harmonics* Y_{lm} and *vector spherical harmonics* $\mathbf{P}_{lm}$, $\mathbf{B}_{lm}$, and $\mathbf{C}_{lm}$ (see e. g. Morse & Feshbach 1953)

$$\mathbf{P}_{lm} = Y_{lm}(\theta, \varphi)\hat{\mathbf{r}}, \quad \mathbf{B}_{lm} = \frac{1}{\sqrt{l(l+1)}}\nabla_\Omega Y_{lm}(\theta, \varphi), \quad \mathbf{C}_{lm} = -\hat{\mathbf{r}} \times \mathbf{B}_{lm}, \tag{9}$$

where ∇_Ω is the angular part of the ∇ operator. We expand the functions $R(\theta,\varphi)$, $V(\theta,\varphi)$, and $F(\theta,\varphi)$ in scalar spherical harmonics,

$$R(\theta,\varphi) = 1 + H_0 + \sum_{lm} R_{lm} Y_{lm}(\theta,\varphi), \tag{10}$$

$$V(\theta,\varphi) = \sum_{lm} V_{lm} Y_{lm}(\theta,\varphi), \qquad F(\theta,\varphi) = \sum_{lm} F_{lm} Y_{lm}(\theta,\varphi), \tag{11}$$

where the sum is for $l \geq 1$ and $-l \leq m \leq l$, and R_{lm}, V_{lm}, and F_{lm} are constant complex coefficients. This expansion makes possible to solve immediately Poisson's equation, obtaining $R_{lm} = -l(l+1)V_{lm}$. Vector quantities like $\nabla_\Omega R$, $\nabla_\Omega V$, etc., are instead expressed as series of vector spherical harmonics, as

$$\nabla_\Omega R = \sum_{lm} \sqrt{l(l+1)} R_{lm} \mathbf{B}_{lm}, \quad \nabla_\Omega V = \sum_{lm} \sqrt{l(l+1)} V_{lm} \mathbf{B}_{lm}, \text{ etc.} \tag{12}$$

The properties of the product of vector spherical harmonics allow non-linear terms to be dealt with in a systematic way in terms of Wigner $3j$ symbols (generalized Clebsh-Gordan coefficients), that can be easily evaluated by Racah's formula (see e.g. Varshalovich, Moskalev, & Khersonskii 1988).

After several manipulations (the details will be given elsewhere), we obtain the equation expressing the condition of radial force balance,

$$H_0 R_{lm} = \sum_{l'm'} \sum_{l''m''} \Theta_{ll'l''}^{mm'm''} A_{l'm'} A_{l''m''}, \tag{13}$$

the equation expressing the condition of tangential force balance,

$$[l(l+1) - 2(1+H_0)] R_{lm} = \sum_{l'm'} \sum_{l''m''} (\Delta_{ll'l''}^{mm'm''} R_{l'm'} R_{l''m''} + \Gamma_{ll'l''}^{mm'm''} A_{l'm'} A_{l''m''}) \tag{14}$$

and the condition for magnetic support,

$$\sum_{lm} |A_{lm}|^2 = 4\pi H_0 (1+H_0), \tag{15}$$

where R_{lm} and $A_{lm} = l(l+1)F_{lm}$ are complex coefficients. The coupling coefficients $\Theta_{ll'l''}^{mm'm''}$, $\Delta_{ll'l''}^{mm'm''}$, and $\Gamma_{ll'l''}^{mm'm''}$ are real, and are equal to zero when the usual triangular conditions on the Wigner $3j$ symbols are not all satisfied. In this way, the representation of the physical variables has been transferred from function space, in terms of θ and φ, to the infinite-dimensional Hilbert space, each vector component now being the coefficient of the corresponding harmonic.

6. Test of the method: the axisymmetric case

The procedure for examining the properties of the spectral equations is to truncate the equations by selecting a finite set of expansion coefficients up to some $l = l_{\max}$, and setting to zero all other coefficients. As a test of the method, we have solved the spectral MHS equations for $m = 0$ and various values of $l_{\max}$,

and compared the results with the axisymmetric solution (singular isothermal toroids) found by LS96. Setting $l_{\max} = 2$, is equivalent to solving only the radial component of the equation of force balance. At this level of approximation, all variables have simple analytical expressions. For increasingly larger values of $l_{\max}$, we solve numerically the resulting nonlinear algebraic system with Newton's method, assuming as a first guess the solution obtained with the previous value of $l_{\max}$. Already for $l_{\max} = 4$, the solution is within 1% of the exact solution obtained with the method of LS96 over a wide range of values of H_0.

6.1. Perturbation analysis

In order to assess the validity of Parker's theorem for models of molecular cloud cores, we investigate whether the class of axisymmetric toroids of LS96 can allow neighbouring 3-D equibria. To this goal, we perform a linearization of the set of nonlinear algebraic equations near the solution determined in the previous section. We consider two kinds of perturbations: (*i*) azimuthally symmetric but non-symmetric with respect to the $\theta = \pi/2$ plane, and (*ii*) symmetric with respect to the $\theta = \pi/2$ plane, but not azimuthally symmetric. In the former case the density perturbation is characterized by l odd and $m = 0$, in the latter by $l + m$ even and $m \neq 0$. The details of the calculation are too lenghty to be given here, and will be published elsewhere.

The results show that the axisymmetric singular isothermal toroids *allow* neighbouring 3-D equilibria of either type (*i*) or (*ii*), for *any* value of H_0, if the perturbation of the density contains harmonics with $l = 1$ (for case *i*) or harmonics with $m = 1$ (for case *ii*). The first case corresponds to a "bending" of the toroid with respect to the original equatorial plane, the second case corresponds to a "stretching" of the toroid on one side, while preserving equatorial symmetry. In the limiting case $H_0 = 0$, where the toroid becomes the unmagnetized singular isothermal sphere, these two distortions are equivalent, and correspond to the small ellipticity limit of the ellipsoidal equilibria found by Medvedev & Narajan (2000). In the limit $H_0 \to \infty$, the toroid becomes a thin disk, and the $m = 1$ distortion reproduces the small ellipticity limit of the sequence of elliptical disklike equilibria found by Galli et al. (2001). Since the ellipsoidal equilibria by Medvedev & Narajan (2000) are not force-free at the central singularity, the same is probably true for the elliptic disklike equilibria found by Galli et al. (2001) and all intermediate toroidal configurations with a $m = 1$ distortion. Very likely, also the "bending" distortion of the toroids results in an equilibrium configuration that is not force-free at the central singularity, but a rigorous proof would be beyond the goals of this paper. We provisionally dismiss both the "$l = 1$ bending" and "$m = 1$ stretching", although the appearance of these permitted distortions of a singular isothermal system deserves further scrutiny.

For arbitrary $m \neq 1$, the perturbation analysis shows that the singular isothermal toroids of LS96 do *not* have neighbouring non-axisymmetric equilibria. This results is in agreement with Parker's theorem, and extends its validity for the first time to the realm of self-gravitating equilibria. Combining this result with the findings of Galli et al. (2001), one is led to the conclusion that the presence of fluid motions (specifically, rotation), rather than the effect of self-gravity, is a crucial ingredient for the occurrence of symmetry breakings in

MHD equilibria. However, the required supermagnetosonic levels of rotation are too high to be of any consequence for molecular cloud cores.

What is then the fate of a quasi-static cloud core and its frozen-in magnetic field, formed by whatever process in a configuration presumably lacking a high degree of symmetry? Parker (1979) argued that realistic magnetic fields with no well defined symmetries must evolve in a genuinely time-dependent way, until all non-symmetric components of the field are destroyed by dissipation and reconnection and the topology becomes symmetric. The numerical simulations of Vainshtein et al. (2000) provide a striking illustration of this process.

The problem with starless cores is that ohmic dissipation times are of the order of $\sim 10^{15}$ yr, and therefore the kind of monotonic relaxation to equilibrium envisaged by Parker can be ruled out. To the extent that cores can be represented as isolated systems devoid of internal sources of energy, these systems must instead undergo weakly damped Alfvèn oscillations around the equilibrium state (say, a non-singular version of the axisymmetric toroids of LS96) with period $\sim L/(c_{\rm s}^2+v_{\rm A}^2)^{1/2}$ (Woltjer 1962), where $v_{\rm A}$ is the Alfvèn speed of the plasma. As we noted in § 2, the deviation of the *average* shape of cloud cores from axisymmetric oblate spheroids is small ($a \approx b$). If we adopt this deviation as a rough measure of nonequilibrium, a simple harmonic oscillator analogy provides an estimate of the average velocity $\langle v\rangle$ of pulsation,

$$\frac{\langle v\rangle}{(c_{\rm s}^2+v_{\rm A}^2)^{1/2}} \approx \frac{\langle a-b\rangle}{a}, \tag{16}$$

which, for $a : b = 1 : 0.8$ (see § 2) gives $\langle v\rangle \approx 0.2\ (c_{\rm s}^2+v_A^2)^{1/2}$. For a typical core with $c_{\rm s} = 0.2$ km s^{-1}, $n({\rm H}_2) = 10^4$ cm^{-3}, and $B = 10\ \mu$G, this implies coherent pulsations with $\langle v\rangle \approx 0.05$ km s^{-1}.

7. Conclusions

Whether or not real cloud cores are triaxial ellipsoids that pulsate because of their intrinsic topological non-equilibrium, as suggested by the theoretical arguments discussed in the previous sections, the material presented in this contribution illustrates well the richness of possible implications concealed in a typical question asked by Frank.

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